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Approximations in Mathematical Morphology and Rough Sets

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Abstract: Mathematical morphology is a general theory in the shapes of images and its transformations. Rough set theory deals with vagueness and uncertainty in the approximation space. The existence of these theories based on operators which are dual in nature. Aim of this paper is to develop an approximation space using set theoretical and topological concepts. For that purpose a new result is developed using binary relations and topological concepts and there by introducing a pre-topological approximation space. This result is applicable in wide range of data mining and image segmentation process.

Keywords: Approximation Space; Mathematical Morphology; Rough Set; Topology; Pre-Topological Approximation Spaces; Pre-Closure and Pre-Interior Operator, Dilation; Erosion.

I. INTRODUCTION

Mathematical morphology developed in the domain of Image processing for the extraction of the properties of images [1]. This theory established as the general theory of shapes and its transformations in images. Mathematical morphology examines the geometrical structure. This procedure results in non-linear image operators which are well suited to explore geometrical and topological structures. Rough set was first introduced to deal with vagueness and uncertainty [2]. Rough set theory has lot of applications in bioinformatics, economics, finance, medicine, software engineering, robotics, power systems, control engineering etc. The main advantage of this theory in data analysis is that it does not need any preliminary or additional information about data-like probability in statistics, grade membership or value of possibility in fuzzy set theory [3]. Generally, in rough set theory no additional information is needed to analyse the data in terms of lower and upper approximations. In this paper Dilation and Erosion are defined by using symmetric structuring elements. This is to show that the lower approximation is equal to erosion and upper approximation is equal to dilation. The existences of the tolerance* relation between the elements of the domain are verified by proving a new result. In this paper, the kurtowski's axiom [9] is used to prove the lower and upper approximations and which is coincide with the interior and closure operators. This introduces the existence of topological approximation spaces which leads to the introduction of pre –topological approximation space in the domain of mathematical morphology. The operators in the pre –topological approximation space is non- idempotent. Hence the basic operator's related morphology can be applied successively to analyse data up to desired degree of accuracy. *A binary relation which is reflexive and symmetric is called a tolerance relation.

II.BASIC OPERATORS IN MATHEMATICAL MORPHOLOGY

The notation B = $\widehat{B}$ = {-b / b ∈ B} is called structuring element with the assumption that it is symmetrical about the origin where $B \subseteq X \subseteq U$ and U is the universal set

A. **Dilation**[1,6,8]
Using Minkowski sum, the dilation of X by a structuring element B is given by
$D_B(X) = X \oplus B = \{x+b / x \in X, b \in B\}$
It is also given by $D_B(X) = \{x \in U / B_x \bigcap X \neq \phi\}$ Where
$B_x = \{x+b / B \in B\}$

B. **Erosion** [1,6,8]
Erosion of X by structuring element B is given by
$E_B(X) = X \ominus B = \{x \in X / X \oplus B \subset X\}$
It is also given by
$E_B(X) = \{x \in U / B_X \subset X\}$
Dilation' and 'Erosion' are the two primary mathematical morphology operators. There are different methods to define these operators.

III.BASIC OPERATORS IN ROUGH SETS

Let R be an arbitrary relation on a universe U and r(x) = {y ∈ U /xRy} be a set of R-related elements of X. For any set X ⊂ U ,a pair of lower and upper approximation is given by $\underline{R}(X)$ and $\overline{R}(X)$ and is defined as

$$\underline{R}(X) = \{x \in U / r(x) \subset X\}$$
$$\overline{R}(X) = \{x \in U / r(x) \cap X \neq \phi\}$$

A .**Rough Set** [2,3,4]
The pair $(\overline{R}(X), \underline{R}(X))$ consisting two crisp sets of lower and upper approximation of the set $X \subseteq U$ is called a rough set.

B. **Approximation Space**
The set U with arbitrary binary relation R is called an approximation space if it satisfies the properties like reflexive, symmetry, transitivity and their combinations

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C. Pawlak Approximation Space [3]

A finite universal set U with equivalence relation $R \subseteq U \times U$ is called Pawlak approximation space. If we consider the structuring element B translated at x of U as disjoint classes, these classes represent elementary details about the image under consideration.

Then for any set $X \subseteq U$, Morphological operations dilation and erosion can be written as

$E_B(X) = \{x \in U / B_x \subseteq X\}$

$D_B(X) = \{x \in U / B_x \bigcap X \neq \phi\}$ and

$E_B(X) \subseteq X \subseteq D_B(X)$

D. Erosion in Terms of Interior

The erosion of a set X w.r.to a structuring element B is the set of all elements of the image which are certainly in X w.r.to B which is considered as the interior of the image under consideration.

E. **Dilation in Terms of Closure**

The dilation of a set X w.r.to a structuring element B is the set of all elements which are possibly in X w.r.to B which acts as the closure of the image.

Remark: 1

According to [5], the operators related to morphology and rough sets are similar. These similarities in properties lead to a new result. This result establishes a connection between the operators in the domain of Morphology and Rough set.

Theorem: 1

If DB(X) and $E_B(X)$ are morphological dilation and erosion of $X \subseteq U$ symmetrical structuring element B containing the origin. Then there exists a tolerance relation R defined by $xRy \Leftrightarrow D_B\{y\}$ with $D_B(X) = \overline{R}(X)$ and $E_B(X) = \underline{R}(X)$

Proof.

Given $xRy \Leftrightarrow D_B\{y\}$

$\Leftrightarrow x \in \{x / B_x \bigcap \{y\} \neq \phi\}$

$\Leftrightarrow y \in B_x$

To prove R is a tolerance relation, we have to prove the following cases

Case(i) R is reflexive

By the definition of translation $B_x = \{x + b / B \in B\}$

Then $x \in B_x \forall x$ (Since B contains the origin) .

$\Rightarrow$ xRx $\forall x$ x. Therefore R is reflexive.

Case(ii) R is symmetric

Also $xRy \Leftrightarrow y \in B_x$

$\Leftrightarrow y - x \in B$

$\Leftrightarrow x - y \in B$ (since B is symmetric)

$\Leftrightarrow x \in B_y$

$\Leftrightarrow yRx$

Therefore R is symmetric. Hence R is a tolerance relation.

From the above steps it is clear that

$B_x = \{y \in U / xRy\} = r(x)$

Therefore from the definitions of dilation and erosion

$D_B(X) = \{x \in U / B_x \bigcap X \neq \phi\}$

$= \{x \in U / r(x) \bigcap X \neq \phi\} = \overline{R}(X)$

$E_B(X) = \{x \in U / B_x \subseteq X\}$

$= \{x \in U / r(x) \subseteq X\} = \underline{R}(X)$

Hence the theorem

IV. TOPOLOGICAL APPROXIMATIONS IN ROUGH SETS

A. **Topological Space**[9,10]

A topological space is a pair $(X, \Im)$ where X is a set and $\Im$ is a family of subsets of X satisfying the following axioms

(i) ϕ, X $\in \Im$

(ii) $\Im$ is closed under arbitrary unions

(iii) $\Im$ is closed under finite intersections

The family $\Im$ is said to be a topology on a set X . Members of $\Im$ are said to open in X or open subsets of X

B. **Closed Set**[9,10]

Let (X, $\Im$) be a topological space. Then a subset A of X is said to be closed in x if its compliment X-A is open in X

C. Closure of a Set [9,10]

The closure of a subset of a topological space is defined as the intersection of all closed subsets containing it.i.e, if A is subset of (X, $\Im$)then its closure is the set $\overline{A}$ is given by $\overline{A} = \cap\{C \subset X / C$ is closed in X ,A$\subset C\}$

D. **Closure Operator** [9,10]

Let $X \subseteq U$ be any set and $P(X)$ be its power set .A closure operator is a mapping $cl : P(X) \rightarrow P(X)$ which satisfies the following axioms

(i) $cl(\phi) = \phi$

(ii) $X \subseteq cl(X) \forall X \subseteq U$

(iii) $cl(X \cup Y) = cl(X) \cup cl(Y)$ for any sets $X, Y \subseteq U$

(iv) $cl[cl(X)] = cl(X)$ (idempotence)

These axioms are called kurtowski axioms [9]for closure operators

E. **Pre-closure Operator**[9,10]

Let $X \subseteq U$ be any set and $P(X)$ be its power set .A pre-closure operator is a mapping $cl_P : P(X) \rightarrow P(X)$ which satisfies the following axioms

(i) $cl_P(\phi) = \phi$

(ii) $X \subseteq cl_P(X) \forall X \subseteq U$

(iii) $cl_P(X \cup Y) = cl_P(X) \cup cl_P(Y)$ for any sets $X, Y \subseteq U$

Therefore a non –idempotent closure operator is called a pre-closure operator

F. **Interior Operator**[9,10]

The dual of the closure operator is called interior operator

$\mathrm{int}(X) = [cl(X^C)]^C$

Let $X \subseteq U$ be any set and $P(X)$ be its power set. An interior operator is a mapping

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$\text{int}: P(X) \to P(X)$ Which satisfies the following axioms?

(i) $\text{int}(U) = U$

(ii) $X \subseteq Int(X) \forall X \subseteq U$

(iii) $\text{int}(X \cap Y) = \text{int}\, X \cap \text{int}\, Y$ for any sets $X, Y \subseteq U$

We can define pre-interior operator by omitting the idempotence of interior operator.

G. **Properties of Lower and Upper Approximations**

Let $X, Y \subseteq U$ and R be an equivalence relation on U Then by [2], the lower approximation $\underline{R}(X)$ and upper approximation $\overline{R}(X)$ satisfy the following properties

(1) $\underline{R}(X) = [\overline{R}(X^C)]^C$

(2) $\underline{R}(U) = U$

(3) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$

(4) $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$

(5) $X \subset Y \Leftrightarrow \underline{R}(X) \subset \underline{R}(Y)$

(6) $\underline{R}(\phi) = \phi$

(7) $\underline{R}(X) \subset X$

(8) $X \subset \underline{R}[\overline{R}(X)]$

(9) $\underline{R}(X) \subseteq \underline{R}[\underline{R}(X)]$

(10) $\overline{R}(X) \subseteq \underline{R}[\overline{R}(X)]$

(11) $\overline{R}(X) = [\underline{R}(X^C]^C$

(12) $\overline{R}(\phi) = \phi$

(13) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$

(14) $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$

(15) $X \subset Y \Rightarrow \overline{R}(X) \subset \overline{R}(Y)$

(16) $\overline{R}(U) = U$

(17) $X \subset \overline{R}(X)$

(18) $\overline{R}[\underline{R}(X)] \subseteq X$

(19) $\overline{R}[\overline{R}(X)] \subseteq \overline{R}(X)$

Theorem: 2.

If R is an equivalence relation on U Then lower approximation is an interior operator and upper approximation is a closure operator satisfying the Kurtowski axioms.

Proof:

The proof of theorem 2 to theorem is analogous to theorem 4 in [2]

It is clear from properties $12,13,15 \& 17$ that the upper approximation satisfies the first three properties of closure operator. Then $X \subset \overline{R}(X) \Rightarrow \overline{R}(X) \subset \overline{R}[\overline{R}(X)]$ By property 19, $\overline{R}[\overline{R}(X)] \subseteq \overline{R}(X)$ .Hence $\overline{R}[\overline{R}(X)] = \overline{R}(X)$.

This implies $\overline{R}$ satisfies all the properties of closure operator. By dual property 1, the lower approximation Coincides with interior operator and hence the theorem.

Theorem: 3

Let $X, Y \subseteq U$,R be an equivalence relation on U , $\overline{R}(X)$ be an upper approximation of X satisfying Kurtowskiaxioms.

Then there exists a topology $\Im$ on U such that $\overline{R}(X)$ coincides with the closure operator associated with $\Im$

Proof:

It is clear from theorem 2 ; the upper approximation $\overline{R}$ satisfies all the properties of closure operator. i.e,

(i) $\overline{R}(\phi) = \phi$

(ii) $X \subset \overline{R}(X)$

(iii) $\overline{R}[\overline{R}(X)] = \overline{R}(X)$

(iv) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$, for any sets $X, Y \subseteq U$

Case (i), existence of a topology $\Im on U$

Let $C = \{X \subseteq U / \overline{R}(X) = X\}$

By $(i) \& (ii)$ $\overline{R}(\phi) = \phi \Rightarrow \phi \in C$

and $U \subseteq \overline{R}(U)$.........................(1)

Also $\overline{R}: U \to U$

then $\overline{R}(U) \subseteq U$..........................(2)

From (1) and (2) $\overline{R}(U) = U \Rightarrow U \in C$

Let $C_i \in C for i = 1,2,3,$.............

By (ii) $\bigcap_i C_i \subseteq \overline{R}(\bigcap_i C_i)$.....................(3)

Also $\bigcap_i C_i \subseteq C_I$ for all i

$\overline{R}(\bigcap_i C_i) \subseteq \overline{R}(C_i)\ \forall i$

$\overline{R}(\bigcap_i C_i) \subseteq \bigcap_i \overline{R}(C_i)$

$\overline{R}(\bigcap_i C_i) \subseteq \bigcap_i C_i$ ....(4)

From (3) & (4)

$\overline{R}(\bigcap_i C_i) = \bigcap_i C_i \Rightarrow \bigcap_i C_i \in$ C

Also by (iv), $\overline{R}\,(\bigcup_{i=1}^{n} C_i) = \bigcup_{i=1}^{n} \overline{R}\,(C_i)$

$= \bigcup_{i=1}^{n} C_i \Rightarrow \bigcup_{i=1}^{n} C_i \in$ C

$\therefore$ If $\Im = \{X^C / X \in C\}$ Then $\Im$ is a topology on U .

Case (ii), to prove that the closure operator associated with $\Im$ coincides with $\overline{R}$

Let $X \subseteq U$, then closure of X , $cl(X)$ is the intersection of all closed subsets of U containing X. But by the nature of C , the closed subsets of U are fixed points of $\overline{R}$

Hence $cl(X) = \bigcap\{Y \subset U$ / X$\subset Y, \overline{R}$ (Y)= Y}

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$X \subset Y \Rightarrow X \cup Y = Y$

$\Rightarrow \overline{R}(X \cup Y) = \overline{R}(Y)$

$\Rightarrow \overline{R}(X) \cup \overline{R}(Y) = \overline{R}(Y)$

$\Rightarrow \overline{R}(X) \subset \overline{R}(Y)$

$\therefore$ If $X \subset Y, \overline{R}(Y) = Y$ then $\overline{R}(X) \subset Y$

But cl(X) is the intersection of all such Y's. Therefore, $\overline{R}(X) \subset cl(X)$

By (iii), $\overline{R}[\overline{R}(X)] = \overline{R}(X) \Rightarrow \overline{R}(X) \in C \Rightarrow \overline{R}(X)$ is closed By (ii) $X \subset \overline{R}(X)$ for all X and $cl(X)$ is the closed set containing X.

Hence $cl(X)$ is the smallest member of C containing X.

Then $\overline{R}(X) \in C \Rightarrow \overline{R}(X) = cl(X)$

Hence the theorem.

Theorem: 4

Let $X \subseteq U$,R be an equivalence relation on U , $\underline{R}(X)$ be a lower approximation of X satisfying Kurtowski axioms.. Then there exists a topology $\Im$ on U such that $\underline{R}(X)$ coincides with the interior operator associated with $\Im$

Similar proof can be done as that of theorem 3

Theorem: 5

Let $X \subseteq U$,R be an equivalence relation on U , $\overline{R}(X)$, $\underline{R}(X)$ be an upper and lower approximation of X satisfying Kurtowski axioms for closure and interior operators respectively . Then there exists a unique topology $\Im$ on U such that $\overline{R}(X)$ coincides with closure operator and $\underline{R}(X)$ coincides with the interior operator associated with $\Im$

Remark:2

Theorem 5 introduces the notion of topological approximation space and its existence is established by proving theorems 3,4 & 5. So that (U, R) is called topological approximation space. The morphological operators erosion and dilation are not idempotent. But these operators satisfy the other 3 axioms of a closure operator. Therefore pre-topological approximation spaces is introducing by using erosion and dilation. This leads to the following theorem.

Theorem :6

Let $X \subseteq U$ and $D_B(X)$ be the dilation of X by a structuring element B. Then there exists a topology $\Im_P$ on U

Proof:

By the definition of dilation $D_B(X) = \{x \in U / B_x \cap X \neq \phi\}$

Then clearly (i) $D_B(\phi) = \phi$

(ii) $X \subseteq D_B(X)$ for any $X \subseteq U$

(iii) $D_B(X \cup Y) = \{z \in U / B_z \cap (X \cup Y) \neq \phi\}$

$= \{z \in U / (B_Z \cap X) \cup (B_Z \cap Y) \neq \phi\}$

$= \{z \in U / (B_Z \cap X) \neq \phi\} \cup \{z \in U / B_z \cap Y) \neq \phi\}$

$= D_B(X) \cup D_B(Y)$ for any $X, Y \subseteq U$

Now to prove there exists a topology on U .

For, consider set of all fixed points of U with respect to the operator D_B

Let M={ $X \subseteq U / D_B(X) = X$ }

Then by (i) $D_B(\phi) = \phi \Rightarrow \phi \in M$

By(ii) $U \subseteq D_B(U)$ ……………………….(1)

Also $D_B : U \to U$ then $D_B(U) \subseteq U$ …….(2)

From (1) & (2) $D_B(U) = U \Rightarrow U \in M$

Let $M_i \in M$ for i= 1,2,3,…….

Then $D_B(M_i) = M_i \; \forall i$

By $\bigcap_i M_i \subseteq D_B(\bigcap_i M_i)$…………………………(3)

Also $(\bigcap_i M_i) \subseteq M_i \forall i$

$D_B(\bigcap_i M_i) \subseteq D_B(M_i) \; \forall i$

$D_B(\bigcap_i M_i) \subseteq \bigcap_i D_B(M_i)$

$D_B(\bigcap_i M_i) \subseteq \bigcap_i M_i$ …………………………(4)

From (3) & (4)

$D_B(\bigcap_i M_i) = \bigcap_i M_i \Rightarrow \bigcap_i M_i \in M$ Also by (iv),

$D_B(\bigcup_{i=1}^n M_i) = \bigcup_{i=1}^n D_B(M_i) = \bigcup_{i=1}^n M_i \Rightarrow \bigcup_{i=1}^n M_i \in M$

$\therefore$ If $\Im_P = \{X^C / X \in M\}$ = $\{X^C / X \in M\}$Then $\Im_P$ is a topology on U .

Hence the theorem

Remark: 3

Here dilation operator satisfies the properties of a pre-closure operator and it induces a topology on U Therefore $(U, \Im_P)$ is topological space .This topological space is a pre-topological approximation space in morphology.

V. CONCLUSION

Data mining technology provides a new thought for organising and managing tremendous data. Rough sets and topology of rough sets constitute a consistency base for data mining and one of the important tool for knowledge discovery and rule extraction according to ref [7].If we use symmetrical structuring element for defining dilation and erosion, by theorem 1 rough set operations can be replaced by morphological operations through a relation which is reflexive and symmetric. The pre-topological concept derived in the domain of mathematical morphology is similar to the topology of rough sets. The only difference is pre-topological space is non idempotent. So it suit best to non-idempotent patterns of data. Therefore the emergence of pre-topological approximation space can contribute new methods in data mining. Also region growing is one of the procedure used in image segmentation .It is a procedure that groups pixels or sub regions into larger regions based on pre-defined criteria for growth [8]. Dilation being apretopological closure operator which is non idempotent it can be applied

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successively on images. Hence it will be suitable for image segmentation by region growing. Also the connection between morphology and rough sets can develop new methods in these areas.

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